

New correction theorems in the light of a weighted Littlewood–Paley–Rubio de Francia inequality

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Abstract

We prove the following correction theorem: every function f on the circumference \mathbb{T} that is bounded by the α_1 -weight w (this means that $Mw^2 \leq Cw^2$) can be modified on a set e with $\int_e w \leq \varepsilon$ so that its quadratic function built up from arbitrary sequence of nonintersecting intervals in \mathbb{Z} will not exceed $C \log \frac{1}{\varepsilon} w$.

1 Introduction

Correction theorems assert that an arbitrary measurable function can be modified on a set of a small measure up to a function with some good properties. Seemingly, the first and the most popular theorem of this type is the classical Lusin theorem about correction up to a continuous function. The next step was D.E.Men'shov's theorem [11] about correction of a bounded measurable function on a set whose measure does not exceed ε up to a function whose partial Fourier sums do not exceed $\frac{C}{\varepsilon}$ uniformly and whose Fourier series converges uniformly. In 1979 S.V.Kislyakov in [10] sharpened the estimate of partial sums up to $C \log \frac{1}{\varepsilon}$ and invented a general method of proving correction theorems. In the paper [5] the estimate was refined.

We consider the circumference equipped with some weight $a(x)$, $x \in \mathbb{T}$, as our general measure space. Similar statements hold for the line, but a slight change of technical details is needed. Now we turn to formal presentation, but first we need some definitions.

1.1 Definitions

First, the Muckenhoupt conditions A_p , $1 \leq p \leq \infty$, will play a significant role in what follows. For every number p , $1 < p < \infty$, this condition can be written as follows:

$$A_p : \sup_I \left(\frac{1}{|I|} \int_I w \right) \left(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} \right)^{p-1} < \infty. \quad (1)$$

If $p = 1$, this condition will turn into $Mw \leq Cw$ with some constant C , where M is the Hardy-Littlewood maximal operator. We say that the weight satisfies the condition A_∞ if it satisfies A_p for some p . The theory of weights that obey such conditions can be found in the book [1]. We also need a more sophisticated condition, α_p , $1 \leq p \leq 2$, which was introduced in the paper [2]. Specially, for $1 < p < 2$ a weight w satisfies α_p if

$$\alpha_p : \sup_I \left(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} \right)^{p-1} \left(\frac{1}{|I|} \int_I w^{\frac{2}{2-p}} \right)^{\frac{2-p}{2}} < \infty. \quad (2)$$

The supremum is taken over the set of all arcs of the circumference. It is easy to see that this condition is equalent to $w^{-\frac{1}{p-1}} \in A_{\frac{p'}{2}}$, or similary, $w^{\frac{2}{2-p}} \in A_{\frac{p}{2-p}}$, where p' is the exponent conjugate to p . The condition α_p can be extended to the border cases of $p = 1$ or $p = 2$ by passing to the limit, and α_1 and α_2 read as follows:

$$\alpha_1 : w^2 \in A_1; \quad \alpha_2 : w^{-1} \in A_1. \quad (3)$$

We see that A_p follows from α_p . Indeed, if $w \in \alpha_p$, then $w^{-\frac{1}{p-1}} \in A_{\frac{p'}{2}}$. Since the Muckenhoupt classes increase as the index increases, we also have $w^{-\frac{1}{p-1}} \in A_{p'}$, which is equalent to the inclusion $w \in A_p$. The case of $p = 1$ can be obtained with the help of the Cauchy-Schwarz inequality.

Second, we need the concept of the quadratic function σ . Let $\Delta_j, j \in \mathbb{N}$, be a family of disjoint segments in \mathbb{Z} . For each of them we introduce the corresponding Fourier multiplier M_{Δ_k} with the help of the following formula:

$$M_{\Delta_k}(f) = (\chi_{\Delta_k}(\xi) \hat{f}(\xi))^\sim. \quad (4)$$

The formula is consistent even if f is a distribution, and a fortiori if $f \in L^1(\mathbb{T})$. Now we can form the quadratic function:

$$\sigma f(x) = \left(\sum_{k \in \mathbb{N}} |(M_{\Delta_k} f)(x)|^2 \right)^{\frac{1}{2}}. \quad (5)$$

It was proved in the fundamental paper [3] that $\|\sigma f\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$ for all $w \in A_{\frac{p}{2}}$ when $2 < p < \infty$. We will mostly use the result of the paper [2], which is somehow dual to the previous one and can be written as follows.

Suppose $1 < p < 2$, $0 < r < p$ and $w \in \alpha_p$; let f_k be a sequence of summable functions such that $\text{supp } \hat{f}_k \subset \Delta_k$. Then the following inequality holds:

$$\|\sum_k f_k\|_{L^r(w)} \leq B_r \left(\sum_k |f_k|^2 \right)^{\frac{1}{2}} \|1\|_{L^r(w)},$$

where B_r does not depend on $\{f_k\}$ (it only depends on the constant in the α_p -estimate of w and r).

This theorem still holds for the cases of $p = 1, p = 2$, they are Corollaries 1, 2 in [2].

1.2 Statement of the main result

Now we are ready to formulate the main result.

Theorem 1. *Suppose a weight a satisfies the A_∞ condition, and a weight w satisfies the α_1 condition. Let f be a measurable function such that $|f| \leq w$. Then for every ε , $0 < \varepsilon < 1$, there exists a function g such that $|g| + |f - g| = |f|$ and the following inequalities hold:*

- 1) $\int_{\{f \neq g\}} a \leq \varepsilon \int_{\mathbb{T}} \left| \frac{f}{w} \right| a,$
- 2) $\sigma g \leq C(a, w)(1 + |\log(\varepsilon)|)w.$

From the condition $|g| + |f - g| = |f|$, it follows that correction is done by multiplying the initial function by some real nonnegative function ϕ whose values do not exceed one. The first inequality estimates the measure of the set where we correct the function. For example, if we take $w = a$, the measure of this set will be estimated by the Lebesgue L^1 -norm of f . The second inequality gives a pointwise estimate of the quadratic function in terms of the weight w . For example, we can try to make the weight w sufficiently small (but it should still be separated from zero, otherwise it will not satisfy the α_1 -condition) on some set. In the fourth section, we will discuss special consequences of Theorem 1 in detail. This theorem looks like Theorem 2' in [4]; in a way it is a generalization of that theorem, because only special sequences of disjoint intervals were involved there, but we have an arbitrary one. On the other hand, we should pay for such a generality and the price is the condition on weight w , A_1 turned into α_1 which, as we know, is stronger. We also mention that in [4] the logarithm in the estimate was squared, in our formula it is not.

We are going to prove this theorem via the general method of obtaining correction theorems, which was described in [5]. We will need a weak $(1, 1)$ -type inequality for some operator, it will be stated in the next subsection.

1.3 An inequality

Suppose μ is a measure, then we will denote by $L^p(l^2, \mu)$ the space of functions with values in l^2 that are summable in the p -th power with respect to μ . Consider the operator T defined on the set of finite sequences of trigonometric polynomials by the following formula:

$$T(\{f_j\}) = \sum_j M_{\Delta_j} f_j. \quad (6)$$

We are going to use another operator T_u , intertwined with T with the help of multiplication by u . Specifically

$$T_u(\{f_j\}) = u^{-1} T(\{u f_j\}). \quad (7)$$

Now we can formulate the second result of this paper.

Theorem 2. *Suppose a weight a satisfies the classical condition A_∞ , a weight w satisfies condition α_1 , see the first section. Let $u = \frac{a}{w}$. Then the operator T_u defined by formulas (6), (7) is continuous from $L^1(l^2, a)$ to $L^{1,\infty}(a)$.*

Strictly speaking, the statement needs further explanations, because we have defined the operator T on the set of trigonometric polynomials, but now we apply it to some other functions. But as usual, it will be seen from the proof that everything is consistent. This theorem looks like theorem 4 in [4], but in that theorem instead of an operator T there was a singular integral operator. T is not an operator of that type, though it can be obtained as a composition of singular integral operators with somewhat nonstandard conditions on the kernel. Of course, the weak $(1, 1)$ -type conditions could have been destroyed under composition, fortunately, this does not happen.

Some words are in order about the operation of multiplication or division by u . An isometry between $L^1(a)$ and $L^1(w)$ is established in this way, both in the case of scalar-valued and l^2 -valued functions. However, the operation fails to establish an isometry between the corresponding weighted Lorentz spaces $L^{1,\infty}$.

Before we turn to the proofs, we should make three small remarks. First, during the proof we assume that all sequences of functions are finite. It will allow us not to think about various technical convergence questions. The general case can be obtained by passing to the limit. Second, we assume all segments Δ_k to be contained in \mathbb{Z}_+ , which will allow us to formulate the theorem in terms of analytic

Hardy classes H_A^p . The general case can be obtained by adding the operators built up from the set of positive segments and the set of negative ones. If some of them contains zero, we can consider it by separately and then add it to the reminder.

As has already been mentioned, we are going to use Hardy classes. To be honest, the continuity of the operators mentioned above is related sooner to the properties of the Hardy spaces than of those of Lebesgue spaces. We explain our notation. By $H_A^p(l^2, a)$ we denote the analytic Hardy class, which consists of all functions from the Smirnov class with values in l^2 whose boundary values are in $L^p(l^2, a(x)dx)$. We will often identify functions belonging to such classes with their boundary values. We will also use the $H_A^{1,\infty}(l^2, a)$ class, we think of it as of the closure of the set of finite sequences of analytic trigonometric polynomials in $L^{1,\infty}(l^2, a)$. This definition is nonstandard, usually $H_A^{1,\infty}(l^2, a)$ is defined as the intersection of $L^{1,\infty}(l^2, a)$ with the Smirnov class. For the L^p -norm, the definitions are equivalent when $p \geq 1$, but for the $L^{1,\infty}$ -quasinorm the equivalence fails. For our purposes it will be more convenient to use the definition with trigonometric polynomials.

We are also going to use some interpolation technique to prove the second theorem. For the reader who is not familiar with it we can advice the book [8]; we also recall the notion of K -closedness and its relation with interpolation. Let (X_0, X_1) be a compatible couple of quasi-Banach spaces, and let Y_0 and Y_1 be closed subspaces of X_0 and X_1 , respectively. The couple (Y_0, Y_1) is said to be K -closed in (X_0, X_1) if for every $y \in Y_0 + Y_1$ and a decomposition $y = x_0 + x_1, x_i \in X_i (i = 0, 1)$ there exists another decomposition $y = y_1 + y_2$, where $y_i \in Y_i$ and $\|y_i\|_{Y_i} \leq C\|x_i\|_{X_i}$. It is easy to see that if (Y_0, Y_1) is K -closed in (X_0, X_1) , then

$$(Y_0, Y_1)_{\theta, q} = (Y_1 + Y_2) \cap (X_0, X_1)_{\theta, q}.$$

See [7] on the concept of K -closedness, its role in interpolation of Hardy spaces.

First, we will prove Theorem 2, second, derive Theorem 1 from it, and then discuss the meaning of these results. We turn to the proof.

2 Proof of the second theorem

The main ideas of the proof are similar to those employed in [6]. We are going to represent T_u as a composition of a finite number of operators \tilde{T}_u with certain singular integral operators \tilde{T} . We preface this by a lemma which is not related to the similarity transformation $\tilde{T} \mapsto u^{-1}\tilde{T}u$ itself, but plays a significant role in the proof.

2.1 The weight mixing lemma

Lemma 1. *Suppose $w \in \alpha_q, a \in A_\infty, 1 < q < 2$. Then there is $\delta, 1 > \delta > 0$, such that for all t in the interval $[1 - \delta, 1)$ there exists r in $(1, 2)$ such that the weight $w^t a^{1-t}$ satisfies the condition α_r .*

Proof. We must estimate the quantity:

$$\left(\frac{1}{|I|} \int_I w^{-\frac{t}{r-1}} a^{\frac{t-1}{r-1}} \right)^{r-1} \left(\frac{1}{|I|} \int_I w^{\frac{2t}{2-r}} a^{\frac{2-2t}{2-r}} \right)^{\frac{2-r}{2}},$$

where r is to be chosen.

Since $a \in A_\infty$, there exists p such that $a \in A_p$. Therefore, by the Jones factorization theorem (see [1], Chapter 5), there exist $a_1, a_2 \in A_1$ such that $a = a_1 a_2^{1-p}$. Now we substitute this new representation for a in the formula and rewrite it in a bit different manner:

$$\begin{aligned} & \left(\frac{1}{|I|} \int_I \frac{1}{w^{\frac{t}{r-1}}} \frac{1}{a_1^{\frac{1-t}{r-1}}} a_2^{\frac{(1-t)(p-1)}{r-1}} \right)^{r-1} \left(\frac{1}{|I|} \int_I w^{\frac{2t}{2-r}} a_1^{\frac{2-2t}{2-r}} \frac{1}{a_2^{\frac{(2-2t)(p-1)}{2-r}}} \right)^{\frac{2-r}{2}} \leq \\ & \frac{1}{\text{essinf}_I a_1^{1-t}} \frac{1}{\text{essinf}_I a_2^{(p-1)(1-t)}} \left(\frac{1}{|I|} \int_I \frac{a_2^{\frac{(p-1)(1-t)}{r-1}}}{w^{\frac{t}{r-1}}} \right)^{r-1} \left(\frac{1}{|I|} \int_I w^{\frac{2t}{2-r}} a_1^{\frac{2-2t}{2-r}} \right)^{\frac{2-r}{2}}. \end{aligned}$$

We estimate each integral separately. We use the standard Hölder inequality with the exponents $\frac{r-1}{(1-t)(p-1)}, \frac{r-1}{r-1-(1-t)(p-1)}$ for the first integral:

$$\left(\frac{1}{|I|} \int_I \frac{a_2^{\frac{(p-1)(1-t)}{r-1}}}{w^{\frac{t}{r-1}}} \right)^{r-1} \leq \left(\frac{1}{|I|} \int_I a_2 \right)^{(1-t)(p-1)} \left(\frac{1}{|I|} \int_I \frac{1}{w^{\frac{t}{r-1-(1-t)(p-1)}}} \right)^{r-1-(1-t)(p-1)}.$$

To use the Hölder inequality, we need that $\frac{r-1}{(1-t)(p-1)} \geq 1$. We remember this condition. Now we use the Hölder inequality with the exponents $\frac{2-r}{2-2t}, \frac{2-r}{2t-r}$ for the second integral:

$$\left(\frac{1}{|I|} \int_I w^{\frac{2t}{2-r}} a_1^{\frac{2-2t}{2-r}} \right)^{\frac{2-r}{2}} \leq \left(\frac{1}{|I|} \int_I w^{\frac{2t}{2t-r}} \right)^{\frac{2t-r}{2}} \left(\frac{1}{|I|} \int_I a_1 \right)^{1-t}.$$

Also, here we should require that $\frac{2-r}{2-2t} \geq 1$.

So, we see that the contribution of the weights a_1 and a_2 to the formula can be estimated by their A_1 -constants in the powers $1-t$ and $(p-1)(1-t)$, respectively. It only remains to estimate the following:

$$\left(\frac{1}{|I|} \int_I \frac{1}{w^{\frac{t}{r-1-(1-t)(p-1)}}}\right)^{r-1-(1-t)(p-1)} \left(\frac{1}{|I|} \int_I w^{\frac{2t}{2t-r}}\right)^{\frac{2t-r}{2}}.$$

Set $r = tq$. If t is sufficiently close to 1, then r is also in $(1, 2)$, therefore, this specification for r is permitted. So we can rewrite the above expression in the form

$$\left(\frac{1}{|I|} \int_I \frac{1}{w^{\frac{t}{tq-1-(1-t)(p-1)}}}\right)^{tq-1-(1-t)(p-1)} \left(\frac{1}{|I|} \int_I w^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}t}.$$

Now, since $w \in \alpha_q$, we can conclude that $w^{-\frac{1}{q-1}} \in A_{\frac{q}{2}}$. Therefore the reverse Hölder inequality is valid for the weight $w^{-\frac{1}{q-1}}$ for some s . Also we note that $\lim_{t \rightarrow 1} \frac{t}{tq-1-(1-t)(p-1)} = \frac{1}{q-1}$. What is more, this value is greater than $\frac{1}{q-1}$. So, for all $t < 1$ in some neighbourhood of 1 we can write the following estimate:

$$\left(\frac{1}{|I|} \int_I \frac{1}{w^{\frac{t}{tq-1-(1-t)(p-1)}}}\right)^{tq-1-(1-t)(p-1)} \leq c \left(\frac{1}{|I|} \int_I w^{\frac{-1}{q-1}}\right)^{t(q-1)}.$$

As a result, after substituting this estimate in the previous one, we get exactly the α_q condition for w , raised to the power t . We also have to check two remembered inequalities. First, the number $\frac{r-1}{(1-t)(p-1)} = \frac{tq-1}{(1-t)(p-1)}$ must be less than one. As $t \rightarrow 1$, this value goes to infinity and, eventually, will exceed one. Second, the number $\frac{2-r}{2-2t} = \frac{2-tq}{2-2t}$ must be greater than one. This value goes to infinity too, so finally the required inequality follows. We have proved the lemma. \square

Now we formulate Lemma 2. Its statement is quite similar to the first lemma, the proofs' difference is only in that the reverse Hölder inequality is applied to another term in brackets. So, we omit it.

Lemma 2. *Suppose $w \in \alpha_q, a \in A_\infty, 1 < q < 2$. Then there exists $\delta, 1 > \delta > 0$, such that for all t in the interval $(1, 1 + \delta]$ there exists $r \in (t, 2)$ such that the weight $w^t a^{1-t}$ satisfies the α_r condition.*

It should be mentioned that the proof of Lemma 1 shows that for t sufficiently close to 1, the weight $w^t a^{1-t}$ satisfies the condition α_{tq} , the same is true for Lemma 2.

Corollary 1. *Both lemmas remain true for $q = 1$.*

We can argue in the following manner: if the weight w satisfies the α_1 condition, it also satisfies the $\alpha_{1+\delta}$ condition with some $\delta > 0$. This can be explained as follows: $w \in \alpha_1$ hence, $w^2 \in A_1$, then, by the reverse Hölder inequality, $w^{2+\varepsilon} \in A_1$, therefore $w^{2+\varepsilon} \in A_{1+\varepsilon}$. But this exactly means that $w \in \alpha_{2-\frac{2}{2+\varepsilon}}$. So it satisfies the assumptions of lemmas.

2.2 Auxiliary operators

We now define two operators, S and R , which came from [6], they also played a significant role in [2]. We begin with the operator S . To define this operator, we assume all intervals Δ_j to be of length 2^l , though each l can occur several times. We name the set of those j whose length is equal to 2^k by B_k . Let $\xi \in (0, 1)$ be some number, we think of it as of a number close to one, and let ϕ_k be trigonometric polynomials on the circumference that satisfy the following conditions borrowed from [6] (see conditions (6), (7) respectively in that paper):

$$\hat{\phi}_m(n) = 0 \quad \text{for } n \notin [0, 2^m]; \quad |\hat{\phi}_m(n)| \leq 1; \quad (8)$$

$$|(\phi_m)^{(r)}(e^{i\sigma})| \leq C_{r,u} 2^{(r+1-u)m} \sigma^{-u} \quad \text{for } \sigma \in [-\pi, \pi], u > 1, r \in \mathbb{Z}_+. \quad (9)$$

Here the differentiation is in the variable σ . These polynomials can also be chosen to satisfy an additional condition, namely: $\hat{\phi}_m = 1$ on $[(1-\xi)2^{m-1}, (1+\xi)2^{m-1}]$. The construction of such polynomials was discussed in [6] in detail. Let $\{h_j\} \in H_A^p(l^2, w)$, then we can define

$$S(h)(x) = \sum_k \sum_{j \in B_k} e^{ia_j x} (h_j * \phi_k)(x). \quad (10)$$

The convolution is well defined, because ϕ_k lies in the Schwarz class and its Fourier transform has compact support, furthermore, we remind the reader that we have agreed to think that the set of the intervals is finite, therefore, the sum is finite too. We can also assume that $p \neq 1$, because, as it has been mentioned, if $w \in \alpha_1$, then $w \in \alpha_{1+\varepsilon}$. Therefore for all p , by Lemma 2 in [2], S is continuous from $H_A^r(l^2, w)$ to $L^r(w)$ when $0 < r < p$ (in [2] everything happened on the line, but for the circumference the arguments are much the same).

The operator R is defined with the help of a family of trigonometric polynomials, namely, let A be some number greater than one. Then there exist polynomials $\beta_j, j > 0$ such that the following two conditions are satisfied:

$$\hat{\beta}_j \geq 0, \sum_j \hat{\beta}_j = \chi_{\mathbb{Z} \setminus \{0\}}, \text{spec } \beta_j \subset [A^{j-1}, A^{j+1}], \quad (11)$$

$$\forall r \in \mathbb{R}_+ \quad R \in \mathfrak{L}(H^r(l^2, w) \rightarrow H^r(l^2, w)), R(\{f_k\}_k) = \{f_k * \beta_j\}_{k,j}. \quad (12)$$

In fact, in this wording (but without weight, i.e., $w = 1$), this statement appeared in [6], named Lemma 1, in [2] it was redesigned slightly and adjusted to the line, it was named Lemma 3 there. In [2] r was not arbitrary, but only from 0 to p if $w \in \alpha_p$.

However, in the present paper we will deal not with the operators S and R directly, but with S_u and R_u , consequently, we want to know that the latter two are continuous. We will not have to invent something new, the usual change of density works. Nevertheless, we will have to interpolate over an unusual scale of spaces to achieve the continuity on the Lorentz class.

2.3 Continuity after a density change

We notice that $f \leftrightarrow uf$ is an isometric bijection between $L^t(l^2, a)$, $L^t(a)$ and $L^t(l^2, w^t a^{1-t})$, $L^t(w^t a^{1-t})$ respectively (t can be smaller than one). We introduce two auxiliary spaces, $E_u^t(l^2) = u^{-1}H_A^t(l^2, w^t a^{1-t})$ and $E_u^t = u^{-1}H_A^t(w^t a^{1-t})$. It is easy to see that $E_u^t, E_u^t(l^2)$ are subspaces of $L^t(a), L^t(l^2, a)$ respectively. The space $E_u^{1,\infty}$ can be defined as the closure in $L_{1,\infty}(a)$ of the set of analytic polynomials divided by u . Obviously, the operators S_u, R_u specified by formula (7) are well defined on a dense subsets of $E_u^t(l^2)$ (trigonometric polynomials divided by u).

Lemma 3. *The operators S_u and R_u are continuous from $E_u^{1,\infty}(l^2)$ to $E_u^{1,\infty}$ and $E_u^{1,\infty}(l^2)$, respectively.*

Proof. We will prove that the operators S_u and R_u are continuous from $E_u^t(l^2)$ to E_u^t and $E_u^t(l^2)$, respectively, for t smaller than one, but lying inside some neighbourhood of it. We notice that by the definition of the spaces $E_u^t(l^2)$, it suffices to prove the continuity of S and R on the spaces $H_A^t(l^2, w^t a^{1-t})$. But by Lemma 1, $w^t a^{1-t} \in \alpha_r$ for some $r \in (1, 2)$. Therefore, we can use Lemmas 2, 3 from [2].

When t is greater than one, but lies in some small neighbourhood of it, the operators S_u and R_u are also continuous from $E_u^t(l^2)$ to E_u^t and $E_u^t(l^2)$ respectively, for similar reasons, one must merely use Lemma 2 instead of Lemma 1.

Finally, we extend the result to the case of ' $t = (1, \infty)$ ' by interpolation. To do this, we notice that the couple $(E_u^t(l^2), E_u^s(l^2))$ is K -closed in $(L^t(l^2, a), L^s(l^2, a))$ if t, s are sufficiently close to 1. Actually, we have an isometry that sends $(E_u^t(l^2), E_u^s(l^2))$ onto $(H_A^t(l^2, w^t a^{1-t}), H_A^s(l^2, w^s a^{1-s}))$. Therefore, we must prove the K -closedness of the last-mentioned couple in $(L^t(l^2, w^t a^{1-t}), L^s(l^2, w^s a^{1-s}))$. To verify this, we use Theorem 3.3 in [7]. It suffices to check that $L^t(l^2, w^t a^{1-t}), L^s(l^2, w^s a^{1-s})$ are BMO -regular lattices, which is true if $\log(w^t a^{1-t}) \in BMO, \log(w^s a^{1-s}) \in$

BMO by Corollary 3.1 in the same paper. But these weights satisfy α_r for some r by Lemmas 1, 2, therefore they satisfy the A_r condition, and the logarithm of such a weight is always in BMO . For scalar-valued spaces everything is the same.

From K -closedness it follows that $(E_u^t(l^2), E_u^s(l^2))_{\theta, \infty} = (E_u^t(l^2) + E_u^s(l^2)) \cap L^{1, \infty}(l^2, a) = u^{-1}N^+ \cap L^{1, \infty}(l^2, a)$ when $\frac{1}{1} = \frac{\theta}{t} + \frac{\theta}{s}$ (N^+ is the Smirnov class, the last identity can be obtained along the lines of a proof of K -closedness in [7]). But surely, the space on the right in this identity includes $E_u^{1, \infty}(l^2)$ as a closed subspace. For the scalar-valued case everything is similar. Consequently, the operators S_u, R_u are continuous on their domains as operators from $E_u^{1, \infty}(l^2)$ to $E_u^{1, \infty}(l^2)$ and $E_u^{1, \infty}$ respectively, so they can be extended by continuity to these spaces, which proves the lemma. \square

2.4 The end of the proof

We can reformulate Theorem 2 as an inequality:

$$\left\| \sum_k u^{-1} M_{\Delta_k}(u f_k) \right\|_{L^{1, \infty}(a)} \leq C \left\| \left(\sum_k |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^1(a)}. \quad (13)$$

We need the following inequality:

$$\left\| u^{-1} \left(\sum_k |M_{\Delta_k}(u f_k)|^2 \right)^{\frac{1}{2}} \right\|_{L^{1, \infty}(a)} \leq C \left\| \left(\sum_k |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^1(a)}, \quad (14)$$

which means that for the projection P defined by the formula $P(\{f_k\}_k) = \{M_{\Delta_k} f_k\}_k$, the corresponding operator P_u is continuous from $L^1(l^2, a)$ to $L^{1, \infty}(l^2, a)$. To prove (14), we observe that $M_{\Delta_k} f_k = e^{2\pi i a_k \cdot} P_+(e^{-2\pi i a_k \cdot} f_k(\cdot)) - e^{2\pi i b_k \cdot} P_+(e^{-2\pi i b_k \cdot} f_k(\cdot))$, where $\Delta_k = [a_k, b_k - 1]$ and P_+ is the Reisz projection. Therefore we have represented P in the form $P = U_1^{-1} P_+ U_1 - U_2^{-1} P_+ U_2$, where U_1, U_2 are operators of multiplication by something unimodular, consequently, isometric operators both on $L^1(l^2, a)$ and on $L^{1, \infty}(l^2, a)$. Obviously, a similar formula holds for P_u . We recall that w is in the α_1 class and, consequently, in A_1 . Therefore, the operator $(P_+)_u$ is continuous from $L^1(l^2, a)$ to $L^{1, \infty}(l^2, a)$ by Theorem 4 in [4]. We see that the desired continuity property holds for P_u , and the inequality is proved.

So we must only prove the following:

$$\left\| \sum_k u^{-1} M_{\Delta_k}(u f_k) \right\|_{L^{1, \infty}(a)} \leq C \left\| u^{-1} \left(\sum_k |M_{\Delta_k}(u f_k)|^2 \right)^{\frac{1}{2}} \right\|_{L^{1, \infty}(a)}. \quad (15)$$

If we denote $g_k = u^{-1} M_{\Delta_k}(u f_k)$, we need to prove that the operator intertwined with the help of multiplication by u with the operator $\{g_k\}_k \mapsto \sum_k g_k$ acts from $E_u^{(1, \infty)}(l^2)$ to $E_u^{(1, \infty)}(l^2)$ (recall that we deal with the case when all intervals Δ_k are contained in \mathbb{Z}_+) and is continuous. During the remaining part of the proof we will be busy with representing this operator as a composition of operators of type R or

S for various sequences of intervals. Then the operator we require ultimately will be represented as a composition of operators like R_u and S_u and will be continuous. To be honest, this procedure was described in detail both in [6] and [2]. We repeat it here for completeness.

Our first purpose is to make our sequence of intervals ‘more regular’. We will use a special ‘cutting’ procedure. We move all our functions f_j so that the left end of Δ_j goes to 1, i.e., we introduce the functions $g_j(x) = e^{-i(a_j-1)x} f_j(x)$, where a_j is the left end of the interval Δ_j . Then we apply the operator R with $A = 2^{\frac{1}{10}}$ to g and then return everything back. As a result, we get a set of functions f_{jk} with more regular spectra, but with the same sum.

We first deal with those j for which the length of Δ_j is at most eleven. We note that for any of such intervals the interval $9\Delta_j$ does not intersect more than 99 other intervals of the partition. Therefore, this set can be split into 100 subsets in such a way that inside each set the spectra of the functions f_j will be separated in a good way (this means that the segments $3\Delta_j$ and $3\Delta_{j'}$ do not intersect when $j \neq j'$, and j, j' are contained in one subset of the partition). After that we apply the operator S to this subsequence (to be more accurate, S will be applied to $e^{-ia_j x} f_j(x)$) and get the part of the entire sum that is generated by these functions. As a result, we have got rid of small intervals.

Each of the remaining functions f_j has been split into several functions f_{jk} . Define the set $A_m, m = [0, \dots, 9], A_m = \{(j, k) | k \equiv_{10} m, f_{jk} \neq 0\}$. That is, we have divided the functions f_{jk} into ten groups in accordance with to the remainder after division k by 10. We note that if we disregard the functions that have the biggest and the biggest but one k for each j , then, in each of the remaining groups, the spectra of functions are separated in a good way, because to the left and to the right of them there are at least two intervals from the other groups (we recall that the spectrum of f_{jk} can intersect the spectrum of $f_{j(k-1)}$, but not of $f_{j(k-2)}$). Therefore we can again apply an operator of type S to the set of functions from each group and get the part of the sum generated by these functions (again, we apply the S -type operator to functions “shifted-to-zero”). As to the functions we have omitted, we proceed similarly, but “in the opposite direction”. Specially, we move the right ends of their intervals to -1 , after that we apply an operator similar to R but generated by antianalytic functions, and then return everything back. Then we do the the same procedure with partition in 10 groups, the only difference is that now we do not need to avoid the small intervals, because the intervals of the partition were big enough and consequently, divided at least into four intervals. Therefore, there was something to the left from the last and the last but one interval, so the last intervals of the new partition are separated in a good way. As a result, their contribution to the entire sum can be rewritten in terms of application of an operator S for some sequence. Now we see that we have

represented the operator $\{g_k\}_k \mapsto \sum_k g_k$ in the desired way, and, thus have proved Theorem 2.

3 The derivation of the first theorem from the second

Consider the set X of bounded measurable functions on the circumference for which $w^{-1}\sigma(wf) \in L^\infty$. We define the norm on this set by the formula:

$$\|f\|_X = \text{esssup}\{|f(\cdot)|, w^{-1}(\sum_k |M_{\Delta_k}(fw)|^2)^{\frac{1}{2}}\}. \quad (16)$$

It is easy to see that X is nonempty. Indeed, an A_1 -weight is separated from zero, therefore, X contains, for example, all functions that can be obtained from trigonometric polynomials via division by w . We still view $(\mathbb{T}, a(x)dx)$ as our main measure space. We are going to use a general theorem from [5] (we alter it a bit, to take into account to the fact that our space has finite measure):

Let a Banach space X of μ -measurable functions satisfy the following two conditions.

A1. The canonical embedding of X into $L^1(\mu)$ is continuous and the unit ball of X is weakly compact in $L^1(\mu)$.

A2. For every $g \in L^\infty$ the functional Φ_g on X defined by the formula $\Phi_g(h) = \int gh d\mu$ satisfies the following inequality:

$$\|g\|_{L^{1,\infty}(\mu)} \leq c\|\Phi_g\|_{X^*},$$

where the constant does not depend on g .

Then for every function F such that $\|F\|_{L^\infty} \leq 1$ and every $\varepsilon, 0 < \varepsilon < 1$, there a function G such that $|G| + |F - G| = |F|$, $\mu(F \neq G) \leq \varepsilon$, $\|G\|_X \leq C(1 + \log \varepsilon^{-1})$

So, we have to check two conditions.

3.1 The first condition

X embeds into L^∞ , consequently, it embeds into $L^1(a)$. We have to check that the unit ball of X is compact in the weak topology of the space $L^1(a)$. We notice that the weak L^1 -convergence on the ball of L^∞ coincides with the weak* convergence in L^∞ regarded as the dual of L^1 ; we will check the compactness in this last topology. We also see that X is a subspace of $(L^\infty \oplus L^\infty(l^2))_\infty$. Indeed, we can define the embedding map $\alpha : X \rightarrow (L^\infty \oplus L^\infty(l^2))_\infty$ with the help of the following formula: $\alpha(h) = (h, \{w^{-1}M_{\Delta_k}(wh)\}_k)$. Therefore our ball B_X becomes the image of the ball αB_X after the canonical projection $(L^\infty \oplus L^\infty(l^2))_\infty$ to the first coordinate.

Thus we are to prove the compactness of αB_X . This set is a subset of the ball of $(L^\infty \oplus L^\infty(l^2))_\infty$, but this space is conjugate to $(L^1(a) \oplus L^1(l^2, a))_1$, its ball is compact by the Alaoglu theorem, so we need only prove the closedness of αB_X viewed as a subset of the ball $(L^\infty \oplus L^\infty(l^2))_\infty$, in other words, we have to prove that if $f_n \rightarrow f$ weakly* in L^∞ and $\|f_n\|_X \leq 1$, then $\|f\|_X \leq 1$, or, again the same, $\|\alpha f\|_{L^\infty(l^2)} \leq 1$. By definition, $\psi \mapsto M_{\Delta_k}(w\psi)$ is a continuous finite rank operator from L^∞ to $C(\mathbb{T})$. Thus, $M_{\Delta_k}(wf_n) \rightarrow M_{\Delta_k}(wf)$ in $C(\mathbb{T})$, consequently, for every N we have the estimate $w^{-1}(\sum_{k=1}^N |M_{\Delta_k}(wf)|)^{\frac{1}{2}} \leq 1$. Passing to the limit in N , we get the desired result. So, we have checked the first condition.

3.2 The second condition

We have to prove the estimate $\|\Phi_g\|_{X^*} \geq c\|g\|_{L^1, \infty(a)}$. X is a closed subspace (as an isometric image) of $(L^\infty \oplus L^\infty(l^2))_\infty$. Now we prove that our functional is continuous in the topology induced on X as on a subspace by the weak* topology of $(L^\infty \oplus L^\infty(l^2))_\infty$, viewed as the dual of $(L^1(a) \oplus L^1(l^2, a))_1$. As we know from the previous subsection, X is a closed subspace in this topology, because its ball is closed (for example, we can use the Banach lemma see [9], addition to the § 5.4). Let $\{h_k\}$ be a sequence in X . Let it converge to some h in the above sense. Then we use the convergence of the first coordinates in $(L^\infty \oplus L^\infty(l^2))_\infty$ and see that $h_k \rightarrow h$ weakly* in L^∞ . But $g \in L^\infty$, $g \in L^1(a)$, therefore $\int h_k g \rightarrow \int h g$. The continuity is proved. As a result, this functional on X can be identified canonically with an element of $(L^1 \oplus L^1(l^2))_1 / \text{Ann } X$ so, we can choose a representative at which the norm is almost attained, i.e., a functional $\tilde{\Phi}$ that extends Φ and satisfies $\tilde{\Phi}((h, \{h_k\}_k)) = \int f h a + \sum_k \int f_k h_k a$, where $(f, \{f_k\}_k) \in (L^1(a) \oplus L^1(l^2, a))_1$, $\|(|f|^2 + \sum_k |f_k|^2)^{\frac{1}{2}}\|_{L^1(a)} \leq \|\Phi\| + \varepsilon$. Therefore, $\Phi(h) = \int f h a + \sum_k \int w^{-1} M_{\Delta_k}(w h) f_k a = \int f h a + \sum_k \int h u^{-1} \overline{M_{\Delta_k}(u \overline{f_k})} a$. Substituting trigonometric polynomials for h , we get $g = f + u^{-1} \sum_k \overline{M_{\Delta_k}(u \overline{f_k})}$.

So we have to prove the following inequality:

$$\|f + u^{-1} \sum_k \overline{M_{\Delta_k}(u \overline{f_k})}\|_{L^1, \infty(a)} \leq c \|(|f|^2 + \sum_k |f_k|^2)^{\frac{1}{2}}\|_{L^1(a)}. \quad (17)$$

Obviously, we can estimate f and the remaining sum separately. But an estimate for f is trivial with the constant one, and the inequality for the sum is precisely Theorem 2 in the form (13). So we have checked the second condition too.

So the conditions of the quoted theorem are fulfilled. But the first theorem is absolutely similar to it, one only have to substitute $F = \frac{f}{w}$ instead of F .

4 Corollaries and a conjecture

4.1 Corollaries to Theorem 1

If we take $w = a = 1$ in the first theorem, we get the following statement.

Theorem 3. *Let f be a measurable function such that $|f| \leq 1$. Then for every $\varepsilon, 0 < \varepsilon < 1$, there exists a function g such that $|g| + |f - g| = |f|$ and the following inequalities hold:*

- 1) $\mu\{f \neq g\} \leq \varepsilon \|f\|_{L^1(\mu)},$
- 2) $\sigma g \leq C(1 + |\log(\varepsilon)|).$

Now let f be an arbitrary measurable function from $L^\infty, |f| \leq 1$. We take $w = (Mf)^\gamma, 0 < \gamma < \frac{1}{2}$. Then $w^2 = (Mf)^{2\gamma} \in A_1$. So, we arrive at the following statement.

Theorem 4. *Let f be a measurable function such that $|f| \leq 1$, and let $a \in A_\infty, \gamma \in (0, \frac{1}{2})$. Then for each $\varepsilon, 0 < \varepsilon < 1$, there exists a function g such that $|g| + |f - g| = |f|$ and the following inequalities hold:*

- 1) $\int_{\{f \neq g\}} a(x) dx \leq \varepsilon \int f(x)^{1-\gamma} a(x) dx,$
- 2) $\sigma g \leq C(1 + |\log(\varepsilon)|)(Mf)^\gamma.$

4.2 Theorem 1 on the line

On the line the first theorem should be formulated in the following way.

Theorem 5. *Let a satisfy A_∞ , let w satisfy α_1 , and let $u = \frac{a}{w}$. Let f be a measurable function with compact support such that $|f| \leq w$. Then for every $\varepsilon, 0 < \varepsilon < 1$, there exists a function g such that $|g| + |f - g| = |f|$ and the following inequalities hold:*

- 1) $\int_{\{f \neq g\}} a \leq \varepsilon \int_{\mathbb{T}} \left| \frac{f}{w} \right| a,$
- 2) $\sigma g \leq C(a, w)(1 + |\log(\varepsilon)|)w.$

The proof is absolutely the same, there is a small difference in a technical detail, because on the line a singular integral operators map L^∞ only to BMO and consequently, its values on L^∞ can be defined only modulo constants. However, if we take the function we are going to correct from $L^\infty \cap L^p$, a singular integral operator will send it to L^p ; for this purpose we impose the compact support condition.

4.3 About the conditions on the weight w

In this subsection we are concerned with several questions about weights in Theorems 1 and 2. The discussion is prefaced by the following lemma.

Lemma 4. *Suppose a weight w satisfies the conditions α_p and A_1 . Then $w \in \alpha_1$*

Proof. We raise the inequality that express the condition A_1 to some power b (to be specified later) and multiply it by the inequality expressing α_p . This results in the following inequality:

$$\left(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}}(x) dx\right)^{p-1} \left(\frac{1}{|I|} \int_I w^{\frac{2}{2-p}}(x) dx\right)^{\frac{2-p}{2}} \left(\frac{1}{|I|} \int_I w(x) dx\right)^b \leq [w]_{\alpha_p} [w]_{A_1} \text{essinf}_I(w^b). \quad (18)$$

Now we apply two Hölder inequalities. The first will be:

$$\left(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}}(x) dx\right)^{(p-1)a} \left(\frac{1}{|I|} \int_I w^{\frac{2}{2-p}}(x) dx\right)^{\frac{2-p}{2}} \geq \quad (19)$$

$$\geq \left(\frac{1}{|I|} \int_I w^{-\frac{a}{c} + \frac{1}{c}}(x) dx\right)^c = \left(\frac{1}{|I|} \int_I w^2(x) dx\right)^c. \quad (20)$$

The constants are: $c = \left(\frac{p-1}{p-1+\frac{2-p}{2}} + 2\right)^{-1}$, $a = \frac{p-1}{p-1+\frac{2-p}{2}}c$. Then the exponents of Hölder inequality are $\frac{c}{a(p-1)}$ and $\frac{2c}{2-p}$, they are conjugate indeed. What is more, we have $-\frac{a}{c} + \frac{1}{c} = 2$, which leads to the second identity.

The second Hölder inequality will look like this (to be precise, this is a Hölder inequality raised to power):

$$\left(\frac{1}{|I|} \int_I w^{-\frac{1}{p-1}}(x) dx\right)^{(p-1)(1-a)} \left(\frac{1}{|I|} \int_I w(x) dx\right)^{1-a} \geq 1. \quad (21)$$

Notice that, since $0 < a < 1$, we can take $b = 1 - a$. Then, after multiplication of the first and the second Hölder inequalities, by using (18) we get the following estimate:

$$\left(\frac{1}{|I|} \int_I w^2(x) dx\right)^c \leq [w]_{\alpha_p} [w]_{A_1} \text{essinf}_I(w^b).$$

We only have to check that $b = 2c$, then it will be exactly the α_1 -condition, raised to the power c . But this is so indeed, and the lemma is proved. \square

It could have been thought that in Theorem 2 one could require $w \in \alpha_p$ rather than $w \in \alpha_1$. Indeed, all the arguments remain valid, except for a small portion about the Reisz projection, between formulas (14) and (15). There we need $w \in A_1$, because the Reisz projection is discontinuous as an operator from $L^1(w)$ to $L^{1,\infty}(w)$ if $w \notin A_1$. This fact is well known, for example, see Proposition 5.4.7 in [1], which corresponds to a nearby situation.

By Lemma 4, these conditions together lead to $w \in \alpha_1$, so we cannot strengthen Theorem 2 in such a way.

4.4 On interpolation

Finally, we state an interesting conjecture, which was partly tackled during the proof of the second theorem and could have shorten it. Namely, we consider the spaces X^p , which are obtained as the closure of the set of finite sequences of trigonometric polynoms that satisfy the conditions $\text{supp } \hat{f}_k \in \Delta_k$, in the topology of $L^p(w)$. Then we suppose the following lemma to be true.

Lemma 5. (*Conjecture*) *Let $p_1 < 1 < p_2$. Then the couple (X^{p_1}, X^{p_2}) is K -closed in (L^{p_1}, L^{p_2}) .*

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